

The Generalized Nagell-Ljunggren Problem: Powers with Repetitive Representations

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Abstract

We consider a natural generalization of the Nagell-Ljunggren equation to the case where the q th power of an integer y , for $q \geq 2$, has a base- b representation that consists of a length- ℓ block of digits repeated n times, where $n \geq 2$. Assuming the *abc* conjecture of Masser and Oesterlé, we completely characterize those triples (q, n, ℓ) for which there are infinitely many solutions b . In all cases predicted by the *abc* conjecture, we are able (without any assumptions) to prove there are indeed infinitely many solutions.

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1 Introduction

Number theorists are often concerned with integer powers, with Fermat’s “last theorem” and Waring’s problem being the two most prominent examples. Another classic problem from number theory is the *Nagell-Ljunggren problem*: for which integers $n, q \geq 2$ does the Diophantine equation

$$y^q = \frac{b^n - 1}{b - 1} \quad (1)$$

have positive integer solutions (y, b) ? See, for example, [36, 37, 31, 32, 39, 44, 28, 23, 10, 11, 14, 13, 45, 1, 9, 8, 12, 34, 15, 35, 7, 24, 27, 30, 2].

On the other hand, in combinatorics on words, repetitions of strings play a large role (e.g., [48, 49, 3]). If w is a word (i.e., a string or block of symbols chosen from a finite alphabet Σ), then by $w \uparrow n$ we mean the concatenation $\overbrace{ww \cdots w}^n$. (This is ordinarily written w^n , but we have chosen a different notation to avoid any possible confusion with the power of an integer.) For example, $(\text{mur}) \uparrow 2 = \text{murmur}$.

In this paper we combine both these definitions of powers and examine the consequences.

In terms of the base- b representation of both sides, the Nagell-Ljunggren equation (1) can be viewed as asking when a power of an integer has base- b representation of the form $1 \uparrow n$ for some integer $n \geq 2$; such a number is sometimes called a “repunit” [50]. An obvious generalization is to consider those powers of integers with base- b representation $a \uparrow n$ for a single digit a ; such a number is sometimes called a “repdigit” [5]. This suggests an obvious further generalization of (1): when does the power of an integer have a base- b representation of the form $w \uparrow n$ for some $n \geq 2$ and some arbitrary word w (of some given nonzero length ℓ)? In this paper we investigate this problem.

Remark 1. A related topic, which we do not examine here, is integer powers that have base- b representations that are palindromes. See, for example, [26, 22, 16].

We introduce some notation. Let $\Sigma_b = \{0, 1, \dots, b - 1\}$. Let $b \geq 2$ be an integer. For an integer $n \geq 0$, we let $(n)_b$ represent the canonical representation of n in base b (that is, the one having no leading zeroes). For a word $w = a_1 a_2 \cdots a_n \in \Sigma_b^n$ we define $[w]_b$ to be $\sum_{1 \leq i \leq n} a_i b^{n-i}$, the value of the word w interpreted as an integer in base b , and we define $|w|$ to be the length of the word w (number of alphabet symbols in it).

Using this notation, we can express the class of equations we are interested in: they are of the form

$$(y^q)_b = w \uparrow n, \quad (2)$$

where $y, q, b, n \geq 2$ and $w \in \Sigma_b^*$. Here we are thinking of q and n as given, and our goal is to determine for which b there exist solutions y and w . Furthermore, we may classify solutions w according to their length $\ell = |w|$.

Alternatively, we can ask about the solutions to the equation

$$y^q = c \frac{b^{n\ell} - 1}{b^\ell - 1}, \quad (3)$$

with $b^{\ell-1} \leq c < b^\ell$. The correspondence of this equation with Eq. (2) is that $w = (c)_b$. The inequality $b^{\ell-1} \leq c < b^\ell$ guarantees that the base- b representation of y^q is indeed an ℓ -digit string that does not start with the digit 0.

Our results can be summarized as follows. We call a triple of integers (q, n, ℓ) for $q, n \geq 2$ and $\ell \geq 1$ *admissible* if either

- $(q, n) = (2, 2)$,
- $(n, \ell) = (2, 1)$, or
- $(q, n, \ell) \in \{(2, 3, 1), (2, 3, 2), (3, 2, 2), (3, 2, 3), (3, 3, 1), (2, 4, 1), (4, 2, 2)\}$.

Otherwise (q, n, ℓ) is *inadmissible*.

Here is our main result:

Theorem 2.

- (a) *Assuming the abc conjecture, there are only finitely many solutions (q, n, ℓ, b, y, c) to (3) such that the triple (q, n, ℓ) is inadmissible.*
- (b) *For each admissible triple (q, n, ℓ) , there are infinitely many solutions (b, y) to the equation $(y^q)_b = w \uparrow n$ for $|w| = \ell$.*

In Section 2 we prove (a) (as Theorem 5) and in Section 3 we prove (b).

One appealing distinction between the Nagell-Ljunggren problem and the variant considered here is that, for fixed n and q , finding solutions to the classical equation 1 amounts to finding the integral points on a single affine curve. Provided that $(q, n) \notin \{(2, 2), (2, 3), (3, 2)\}$, the genus of this curve is positive, so Siegel's theorem implies that it has only finitely many integer points. On the other hand, in the variant considered here, for fixed n , q , and ℓ , finding solutions to Eq. (3) amounts to finding integral points of controlled height on a family of twists of a single curve, which is well known to be a hard problem. Moreover, there is an established literature of using the *abc* conjecture to attack such problems; for example, see [20].

We comment briefly on our representation of words. In some cases, particularly if $b \leq 10$, we write a word as a concatenation of digits. For example, 1234 is a word of length 4. However, if $b > 10$, this becomes infeasible. Therefore, for $b \geq 10$, we write a word using parentheses and commas. For example, $(11, 12, 13, 14)$ is a word of length 4 representing 40034 in base 15.

2 Implications of the *abc* conjecture

Let $\text{rad}(n) = \prod_{p|n} p$ be the radical function, the product of distinct primes dividing n . We recall the *abc* conjecture of Masser and Oesterlé [33, 41], as follows (see, e.g., [47, 38, 6, 21, 42]):

Conjecture 3. For all $\epsilon > 0$, there exists a constant C_ϵ such that for all $a, b, c \in \mathbb{Z}^+$ with $a + b = c$ and $\gcd(a, b) = 1$, we have

$$c \leq C_\epsilon(\text{rad}(abc))^{1+\epsilon}.$$

We will need the following technical lemma. Its purpose will become clear in the proof of Theorem 5. The proof is a straightforward manipulation of inequalities, but we include it for the sake of completeness.

Lemma 4. Suppose that q, n, ℓ are positive integers with $q \geq 2$, $n \geq 2$, and $\ell \geq 1$. Further suppose that (q, n, ℓ) is not an admissible triple. Define

$$F(q, n, \ell) = \frac{24}{25}n\ell - 1 - \frac{n\ell}{q} - \ell.$$

Then $F(q, n, \ell) > 0$.

Proof. If (q, n, ℓ) is not admissible, then either $n \geq 3$ or both $q \geq 3$ and $\ell \geq 2$.

First assume that $n \geq 3$. Then

$$F(q, n, \ell) = n\ell \left(\frac{24}{25} - \frac{1}{q} \right) - 1 - \ell \geq \frac{47}{25}\ell - \frac{3\ell}{q} - 1.$$

This quantity is positive if and only if

$$q \geq \frac{75\ell}{47\ell - 25}.$$

For $\ell \geq 1$, the quantity $\frac{75\ell}{47\ell - 25}$ is strictly less than 2, and $q \geq 2$, so $F(q, n, \ell) > 0$.

Now assume instead that $q \geq 3$ and $\ell \geq 2$. Rearranging the inequality in a different way, we see that $F(q, n, \ell) > 0$ if and only if

$$n \geq \frac{\ell q + q}{\frac{24}{25}\ell q - \ell} = 25 \left(\frac{\ell + 1}{\ell} \right) \left(\frac{q}{24q - 25} \right).$$

This quantity is decreasing in ℓ and increasing in q for all $\ell \geq 1$ and $q \geq 2$, and it is strictly less than $\frac{75}{48}$, which is less than 2. As $n \geq 2$, $F(q, n, \ell)$ is positive in this case as well. \square

Theorem 5. Assume the abc conjecture. There are only finitely many solutions (q, n, ℓ, b, y, c) to the generalized Nagell-Ljunggren equation such that (q, n, ℓ) is an inadmissible triple.

Proof. The equation $(y^q)_b = w \uparrow n$ can be written

$$y^q = c \left(\frac{b^{n\ell} - 1}{b^\ell - 1} \right)$$

for $c \in \mathbb{Z}$ such that $(c)_b = w$. Note that $c \leq b^\ell - 1$, so $y < b^{n\ell/q}$.

Suppose p is a prime that divides $\frac{b^{n\ell}-1}{b^\ell-1}$. Then p divides y^q , and thus p^q divides y^q . Therefore

$$y^q \geq \left(\text{rad} \left(\frac{b^{n\ell}-1}{b^\ell-1} \right) \right)^q \geq \frac{(\text{rad}(b^{n\ell}-1))^q}{(\text{rad}(b^\ell-1))^q},$$

where we have used the obvious inequality $\text{rad}(a/b) \geq \text{rad}(a)/\text{rad}(b)$. So

$$\text{rad}(b^{n\ell}-1) \leq y \text{rad}(b^\ell-1) < yb^\ell < b^{n\ell/q+\ell},$$

using $y < b^{n\ell/q}$ and $\text{rad}(b^\ell-1) < b^\ell$.

Now consider the equation

$$(b^{n\ell}-1) + 1 = b^{n\ell}.$$

By the *abc* conjecture, for all $\epsilon > 0$, there is some positive constant C_ϵ such that

$$b^{n\ell} \leq C_\epsilon (\text{rad}((b^{n\ell}-1)(1)b^{n\ell}))^{1+\epsilon} \leq C_\epsilon (b \text{rad}(b^{n\ell}-1))^{1+\epsilon},$$

using that $b^{n\ell}$ and $b^{n\ell}-1$ are coprime and that $\text{rad}(b^{n\ell}) = \text{rad}(b) \leq b$. We rewrite this inequality as

$$b^{n\ell/(1+\epsilon)-1} \leq C_\epsilon^{1/(1+\epsilon)} \text{rad}(b^{n\ell}-1).$$

Set $C'_\epsilon = C_\epsilon^{1/(1+\epsilon)}$. Combining the upper and lower bounds on $\text{rad}(b^{n\ell}-1)$, we get

$$b^{n\ell/(1+\epsilon)-1} \leq C'_\epsilon b^{n\ell/q+\ell}.$$

Rearranging this, we have

$$b^{n\ell/(1+\epsilon)-1-n\ell/q-\ell} \leq C'_\epsilon,$$

or equivalently,

$$\frac{n\ell}{(1+\epsilon)} - 1 - \frac{n\ell}{q} - \ell \leq \frac{\log(C'_\epsilon)}{\log(b)}. \quad (4)$$

Recall that $y < b^{n\ell/q}$, or equivalently

$$\frac{1}{\log(b)} < \frac{n\ell}{q \log(y)}.$$

Therefore

$$\frac{n\ell}{(1+\epsilon)} - 1 - \frac{n\ell}{q} - \ell \leq \log(C'_\epsilon) \frac{n\ell}{q \log(y)}. \quad (5)$$

In order for the triple (q, n, ℓ) to give rise to a solution of $(y^q)_b = w \uparrow n$, it is necessary that inequalities (4) and (5) are both satisfied. This puts restrictions on b and y , respectively.

From this point forward, fix $\epsilon = \frac{1}{24}$. (Any fixed choice of $\epsilon < \frac{1}{23}$ would work for our purposes.) Let

$$F(q, n, \ell) = \frac{n\ell}{(1+\epsilon)} - 1 - \frac{n\ell}{q} - \ell.$$

It is easy to see that F is increasing in q . We will soon see that F is also increasing in n and ℓ when (q, n, ℓ) is inadmissible.

It can be verified by an explicit calculation that $F(q, n, \ell) < 0$ for all admissible triples (q, n, ℓ) , including the infinite families with $(q, n) = (2, 2)$ and ℓ arbitrary or $(n, \ell) = (2, 1)$ and q arbitrary. By Lemma 4, for every inadmissible triple (q, n, ℓ) we have $F(q, n, \ell) > 0$, so there are only finitely many b that satisfy inequality (4). We will show that for large values of n or ℓ , no bases $b \geq 2$ satisfy (4), and for large values of q , no $y \geq 2$ satisfy (5) (clearly $y = 1$ never gives a solution). Therefore, conditional on the *abc* conjecture, there are only finitely many solutions to the generalized Nagell-Ljunggren equation that come from inadmissible parameters.

First we consider large values of n or ℓ by computing lower bounds on the partial derivatives of F . Assume that (q, n, ℓ) is not admissible, and therefore either $n \geq 3$ and $q \geq 2$ or $n \geq 2$ and $q \geq 3$. Then we have lower bounds on the partial derivatives as follows:

$$\begin{aligned}\frac{\partial F}{\partial n} &= \ell \left(\frac{1}{1+\epsilon} - \frac{1}{q} \right) \geq 1 \left(\frac{1}{1+\epsilon} - \frac{1}{2} \right) = \frac{23}{50} \\ \frac{\partial F}{\partial \ell} &= n \left(\frac{1}{1+\epsilon} - \frac{1}{q} \right) - 1 \geq \min \left(\frac{19}{75}, \frac{19}{50} \right) = \frac{19}{75}\end{aligned}$$

If $n \geq 5$, then we have

$$F(q, n, \ell) \geq F(2, n, 1) \geq \frac{23}{50}(n-5) + F(2, 5, 1) > \frac{23}{50}(n-5),$$

If $\ell \geq 5$, we have

$$\begin{aligned}F(q, n, \ell) &\geq \min(F(3, 2, \ell), F(2, 3, \ell)) \\ &\geq \min \left((\ell-4) \frac{19}{75} + F(3, 2, 4), (\ell-5) \frac{19}{75} + F(2, 3, 3) \right) \\ &> \frac{19}{75}(\ell-4).\end{aligned}$$

Importantly, in the above calculations we have used both that $F(q, n, \ell) > 0$ and that F is increasing in q , n , and ℓ for all inadmissible triples. So $F(q, n, \ell) \rightarrow \infty$ as either $n \rightarrow \infty$ or $\ell \rightarrow \infty$. Thus for large values of either n or ℓ , inequality (4) is not satisfied for any $b \geq 2$, and there are no solutions to $(y^q)_b = w \uparrow n$.

It remains to show that large values of q cannot be used in solutions. First we rewrite inequality (5) as

$$q \left(\frac{1}{1+\epsilon} - \frac{1}{n\ell} - \frac{1}{n} \right) \leq \frac{\log(C'_\epsilon)}{\log(y)} + 1.$$

If (q, n, ℓ) is inadmissible, then either $n \geq 3$ and $\ell \geq 1$ or $n \geq 2$ and $\ell \geq 2$. So

$$\frac{1}{n\ell} + \frac{1}{n} = \frac{1}{n} \left(1 + \frac{1}{\ell} \right) \leq \frac{3}{4}$$

and

$$\frac{21q}{100} = q \left(\frac{24}{25} - \frac{3}{4} \right) \leq \frac{\log(C'_\epsilon)}{\log(y)} + 1 \leq \frac{\log(C'_\epsilon)}{\log(2)} + 1,$$

where we have replaced y with 2, which is the smallest value of y that can be used in a solution. So for inadmissible triples (q, n, ℓ) with large values of q , inequality (5) is not satisfied, and there are no solutions.

We have shown that there are only finitely many inadmissible triples that admit any solutions. By Lemma 4 and inequality (4), there are only finitely many bases b that can appear in a solution corresponding to each such triple, and thus only finitely many solutions for each such triple. So the set of all inadmissible triples contributes in total only finitely many solutions. \square

Remark 6. Shinichi Mochizuki, in a series of papers released in 2016, has recently claimed a proof of the *abc* conjecture. If the proof is ultimately verified, then Theorem 5 will hold unconditionally.

3 Admissible triples

In this section we examine each admissible triple and prove there are infinitely many solutions.

3.1 The case $(q, n) = (2, 2)$

Theorem 7. *For each length $\ell \geq 1$, there are infinitely many $b \geq 2$ such that the equation $(y^2)_b = w \uparrow 2$ has a solution with $|w| = \ell$.*

We need a lemma.

Lemma 8. *For each integer $t \geq 0$ there exist infinitely many integer pairs (p, b) where $p \geq 2$ is prime and $b \geq 2$ such that $b^{2^t} \equiv -1 \pmod{p^2}$. Furthermore, among these pairs there are infinitely many distinct b .*

Proof. By Dirichlet's theorem on primes in arithmetic progressions, there are infinitely many primes $p \equiv 1 \pmod{2^{t+1}}$. The group G of integers modulo p^2 is cyclic, and of order $p(p-1)$. Since $2^{t+1} \mid p-1$, there is an element b of order 2^{t+1} in G . For this element b we have $b^{2^t} \equiv -1 \pmod{p^2}$.

To prove the last claim, note that for each fixed t and fixed b there are only finitely many prime divisors of $b^{2^t} + 1$. If there were only finitely many distinct b among those pairs (p, b) with $b^{2^t} \equiv -1 \pmod{p^2}$, then there would only be, in total, finitely many pairs (p, b) , contradicting what we just proved. \square

Now we can prove Theorem 7.

Proof. Let $\ell = r \cdot 2^t$, where r is odd. By Lemma 8 we know there exist infinitely many p and b such that $b^{2^t} \equiv -1 \pmod{p^2}$. Then $b^\ell = b^{r \cdot 2^t} \equiv -1 \pmod{p^2}$.

Now write $b^\ell + 1 = mp^2$. Then $m^2(b^\ell + 1) = m^2p^2$. Choose $v = \lceil \frac{p}{\sqrt{b}} \rceil$. Then

$$\frac{p}{\sqrt{b}} \leq v \leq \frac{p}{\sqrt{b}} + 1,$$

so

$$\frac{p^2}{b}m \leq mv^2 \leq m \left(\frac{p}{\sqrt{b}} + 1 \right)^2.$$

Hence

$$mv^2 \geq mp^2/b = \frac{b^\ell + 1}{b} \geq b^{\ell-1}.$$

Similarly, if $p \geq 5$, then $\frac{p}{\sqrt{2}} + 1 \leq \frac{p}{1.1}$, so

$$mv^2 \leq m \left(\frac{p}{\sqrt{b}} + 1 \right)^2 \leq m \left(\frac{p}{1.1} \right)^2 \leq mp^2 - 1$$

if $p \geq 5$.

Then $(mvp)^2 = (mv^2)(b^\ell + 1)$. The inequalities obtained above imply that mv^2 in base b is an ℓ -digit number, so the base- b representation of $(mvp)^2$ consists of two copies of $(mv^2)_b$, as desired.

From the second part of the Lemma, we get that there are infinitely many b corresponding to each length ℓ . \square

Example 9. Take $\ell = 12$. Then $r = 3$ and $t = 2$. If $b = 110$ and $p = 17$, then $b^4 \equiv -1 \pmod{17^2}$. Write $b^\ell + 1 = m \cdot p^2$, where $m = 10859613760280276816609$. Let $v = \lceil \frac{p}{\sqrt{b}} \rceil = 2$. Then $mvp = 369226867849529411764706$ and $((mvp)^2)_b = w \uparrow 2$ where

$$w = [1, 57, 52, 15, 108, 52, 57, 94, 1, 57, 52, 16].$$

We now examine this case from a different angle, considering b to be fixed and examining for which pairs (y, w) there are solutions to $(y^2)_b = w \uparrow 2$.

Theorem 10. *For each base $b \geq 2$, the equation*

$$(y^2)_b = w \uparrow 2$$

has infinitely many solutions (y, w) .

First, we need a lemma:

Lemma 11. *For all integers $b \geq 2$, there exists a prime $p \geq 5$ such that b has even order in the multiplicative group of integers modulo p^2 .*

Proof. First observe that if b has even order mod p , then it must have even order mod p^2 .

If there is some prime $p \geq 5$ that divides $b + 1$, then p cannot also divide $b - 1$. Then $b^2 \equiv 1 \pmod{p}$ and $b \not\equiv 1 \pmod{p}$, so b has order 2 mod p , and we are done. Therefore it suffices to prove the Lemma in the case that the primes dividing $b + 1$ are a subset of $\{2, 3\}$.

We aim to show that there is some prime $p \geq 5$ that divides $b^2 + 1$. Then p cannot also divide $b^2 - 1$, and so

$$b^4 - 1 = (b^2 + 1)(b^2 - 1) \equiv 0 \pmod{p}$$

and b has order 4 mod p . Assume, to get a contradiction, that the only possible prime factors of $b^2 + 1$ are 2 and 3.

By the Euclidean algorithm, $\gcd(b^2 + 1, b + 1) = \gcd(1 - b, b + 1) = \gcd(2, b + 1)$, so 2 is the only possible common prime divisor of both $b^2 + 1$ and $b + 1$. In particular, it is not possible that both numbers are divisible by 3. Therefore one of $b + 1$ or $b^2 + 1$ is a power of 2; thus b is odd, and $\gcd(b + 1, b^2 + 1) = 2$. This leaves two possibilities: either $b + 1 = 2^n$ and $b^2 + 1 = 2 \cdot 3^m$, or $b + 1 = 2 \cdot 3^m$ and $b^2 + 1 = 2^n$, for some positive integers n, m .

If $b + 1 = 2^n$, then $b + 1 \equiv 1 \pmod{3}$, so $b \equiv 0 \pmod{3}$. But then $b^2 + 1$ cannot be divisible by 3. So instead we must have $b + 1 = 2 \cdot 3^m$, and

$$2^n = b^2 + 1 = (2 \cdot 3^m - 1)^2 + 1 = 2(2 \cdot 3^{2m} - 2 \cdot 3^m + 1).$$

So 2^n is twice an odd number, and $n = 1$. But then $b = 1$, which is a contradiction. \square

We can now prove Theorem 10:

Proof. Fix b , and let $p \geq 5$ be a prime satisfying the conclusion of Lemma 11. Let the order of b , modulo p^2 , be $e' = 2e$ for some integers $e', e \geq 1$.

First, we claim that for all $b \geq 2$ such a p can be chosen such that there is an integer t with

$$b^{-1/4} \sqrt{p} < t < \sqrt{9p/10}. \quad (6)$$

If $b \geq 16$, then the open interval $(b^{-1/4} \sqrt{p}, \sqrt{9p/10})$ has length > 1 if $p \geq 5$, and hence contains an integer.

If $2 \leq b < 16$, we can use the t and p in the table below:

| b | p | t |
|-----|-----|-----|
| 2 | 5 | 2 |
| 3 | 5 | 2 |
| 4 | 5 | 2 |
| 5 | 7 | 2 |
| 6 | 7 | 2 |
| 7 | 5 | 2 |
| 8 | 5 | 2 |
| 9 | 5 | 2 |
| 10 | 7 | 2 |
| 11 | 13 | 3 |
| 12 | 5 | 2 |
| 13 | 5 | 2 |
| 14 | 5 | 2 |
| 15 | 13 | 3 |

Hence, from (6) we get

$$p^2/b < t^4 < .81p^2$$

and so

$$1/b < t^4/p^2 < .81.$$

Now consider $z = (t^4/p^2)(b^{re} + 1)$ for odd $r \geq 1$. Since b has order $2e \pmod{p^2}$, we must have $b^e \equiv -1 \pmod{p^2}$. Then for odd $r \geq 1$ we have $b^{re} \equiv -1 \pmod{p^2}$, and so $z = \frac{t^4}{p^2}(b^{re} + 1)$ is an integer. From the previous paragraph we have

$$b^{re-1} < (t^4/p^2)b^{re} < (t^4/p^2)(b^{re} + 1) = z,$$

and

$$z = \frac{t^2}{p}(b^{re} + 1) < 0.81(b^{re} + 1) < b^{re},$$

where the very last inequality holds provided $b^{re} \geq 5$. If $b \geq 5$ this inequality holds for all e . For smaller b , we can choose e as follows to ensure $b^{re} \geq 5$:

- if $b = 2$ then $p = 5$ and $e = 10$;
- if $b = 3$ then $p = 5$ and $e = 10$;
- if $b = 4$ then $p = 5$ and $e = 5$.

It follows that the base- b representation of z has exactly re digits. Let $w = (z)_b$. Finally, note that

$$[ww]_b = \frac{t^4}{p^2}(b^{re} + 1)(b^{re} + 1) = \left(\frac{t^2}{p}(b^{re} + 1)\right)^2,$$

so we can take $y = \frac{t^2}{p}(b^{re} + 1)$. \square

Remark 12. For $b = 2$ the solutions y to the equation $(y^2)_2 = w \uparrow 2$ are given by the sequence

$$6, 820, 104391567, 119304648, 858993460, 900719925474100, \dots,$$

which is sequence [A271637](#) in the OEIS [40].

3.2 The case $(n, \ell) = (2, 1)$

This case, where $n = 2$ and $\ell = 1$, is the least interesting of all the cases.

Proposition 13. *The equation $(y^q)_b = w \uparrow 2$, $|w| = 1$, has infinitely many solutions b for each $q \geq 2$.*

Proof. The equation can be rewritten as $y^q = c(b + 1)$ for $1 \leq c < b$. Given q , we can take $c = 1$, $y \geq 2$, and $b = y^q - 1$. \square

3.3 The case $(q, n, \ell) = (2, 3, 1)$

In this section we show

Theorem 14. *There are infinitely many bases b for which the equation $(y^2)_b = w \uparrow 3$ has a solution with $|w| = 1$.*

Proof. We want to show there are infinitely many positive integer solutions to

$$y^2 = c(b^2 + b + 1) \tag{7}$$

with $1 \leq c < b$. We show below that there are infinitely many integral points on the affine curve defined by

$$3y^2 = x^2 + x + 1 \tag{8}$$

with $x > 0$. Taking such a point, we easily obtain a solution to (7) with $c = 3$, namely $(3y)^2 = 3(x^2 + x + 1)$.

We rewrite (8) as a norm equation in the real quadratic field $\mathbb{Q}(\sqrt{3})$. In particular, rearranging terms yields

$$(2x + 1)^2 - 12y^2 = -3,$$

which is equivalent to $N((2x + 1) + 2y\sqrt{3}) = -3$, where N is the norm from $\mathbb{Q}(\sqrt{3})$ to \mathbb{Q} . Running this process in reverse, if $\alpha \in \mathbb{Q}(\sqrt{3})$ has norm -3 and can be written in the form $\alpha = a + b\sqrt{3}$ for positive integers a, b with a odd and b even, then $x = (a - 1)/2$, $y = b/2$ gives an integer point on $3y^2 = x^2 + x + 1$.

The unit group of $\mathbb{Z}[\sqrt{3}]$ (which is the ring of integers of $\mathbb{Q}(\sqrt{3})$) is generated by -1 and the fundamental unit $u = 2 - \sqrt{3}$, which has $N(u) = 1$. If α is any element of the desired form (e.g., $\alpha = 1 + 2\sqrt{3}$), then $\alpha u^{2k} = a_k + b_k\sqrt{3}$ will also have norm -3 . Moreover,

$$u^2 = 7 - 4\sqrt{3} \equiv 1 \pmod{2\mathbb{Z}[\sqrt{3}]},$$

so that $\alpha u^{2k} \equiv \alpha \pmod{2\mathbb{Z}[\sqrt{3}]}$. Thus, a_k is odd and b_k is even for every $k \in \mathbb{Z}$. This gives infinitely many integer solutions to Eq. (8), which, multiplying a_k and b_k by -1 if necessary, we may assume to have $x > 0$. □

Remark 15. A similar class of solutions can be found for any $c > 0$ for which the real quadratic field $\mathbb{Q}(\sqrt{c})$ has an integral element of norm -3 . Another such field is $\mathbb{Q}(\sqrt{7})$, for which the first associated solution is $(49^2)_{18} = [7, 7, 7]$.

3.4 The case $(q, n, \ell) = (2, 3, 2)$

Theorem 16. *There are infinitely many solutions to the case $(q, n, \ell) = (2, 3, 2)$.*

Proof. We would like to find solutions to

$$y^2 = c(b^4 + b^2 + 1) \tag{9}$$

in positive integers b, y, c such that $b \leq c < b^2$. Without loss of generality, any integer solution can be replaced by a positive integer solution.

Notice that $x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1)$, and suppose that (x, y) is a integral point on the curve $3y^2 = x^2 + x + 1$ such that $x^2 - x + 1$ is divisible by 49. Let $c = \frac{3}{49}(x^2 - x + 1)$, so that $3c(x^2 - x + 1)$ is a square. We compute

$$\left(\sqrt{3c(x^2 - x + 1)}y\right)^2 = 3c(x^2 - x + 1)y^2 = c(x^2 - x + 1)(x^2 + x + 1) = c(x^4 + x^2 + 1),$$

which gives a solution to Eq. (9) with $b = x$, as long as $x \leq c < x^2$; this inequality holds provided that $x \geq 18$. We now produce infinitely many integral points on the curve $3y^2 = x^2 + x + 1$ such that $49 \mid x^2 - x + 1$, so this inequality is not an issue.

As in the (2,3,1) case, we rewrite $3y^2 = x^2 + x + 1$ as the norm equation

$$N((2x + 1) + 2y\sqrt{3}) = -3,$$

where N is the norm from $\mathbb{Q}(\sqrt{3})$ to \mathbb{Q} . Again as in the (2,3,1) case, if $\alpha \in \mathbb{Q}(\sqrt{3})$ has norm -3 and can be written as $\alpha = a + b\sqrt{3}$ for positive integers a, b with a odd and b even, then $x = (a - 1)/2$, $y = b/2$ gives an integer point on $3y^2 = x^2 + x + 1$. Observe that the polynomial $x^2 - x + 1$ will be divisible by 49 if and only if either $x \equiv 19$ or $x \equiv 31 \pmod{49}$. If $a = 2x + 1$, this occurs if and only if either $a \equiv 39$ or $a \equiv 63 \pmod{98}$.

Observe that $\alpha = 627 + 362\sqrt{3}$ satisfies the desired congruence conditions. Let $u = 2 - \sqrt{3}$ be the fundamental unit of $\mathbb{Z}[\sqrt{3}]$. As u is a unit, some power u^r will necessarily be congruent to 1 $\pmod{98\mathbb{Z}[\sqrt{3}]}$; an explicit computation shows that $r = 56$ works. Thus, setting $\alpha u^{56k} = a_k + b_k\sqrt{3}$, for every $k \in \mathbb{Z}$ we have $a_k \equiv 39 \pmod{98}$ and that b_k is even. This produces the infinitely many solutions to Eq. (9). The next one, with $k = 1$, is

$$(b, y, c) = (33519770429365238471302383574583401, \\ 19352648480568478024495121554106701, \\ 68790306712490710007811612444611710421528067927390557506093905927147).$$

□

3.5 The case $(q, n, \ell) = (3, 2, 2)$

Theorem 17. *There are infinitely many solutions to the case $(q, n, \ell) = (3, 2, 2)$.*

Proof. The equation we want to solve in positive integers is

$$y^3 = c(b^2 + 1) \quad (10)$$

for $b^2 \leq c < b^3$.

We begin by showing that there are infinitely many integral points (x, y) on the curve

$$2y^2 = x^2 + 1, \quad (11)$$

where without loss of generality, both x and y are positive. Starting with such a point (x, y) and rearranging Eq. (11), we have $(2y)^3 = 4y(x^2 + 1)$, which gives a solution to Eq. (10) with $b = x$ and $c = 4y$. As $y = \sqrt{(x^2 + 1)/2}$, certainly $c \geq x$. The upper bound $c \leq x^2 - 1$ is equivalent to $2\sqrt{2(x^2 + 1)} \leq x^2 - 1$, which can be verified to hold for all $x \geq 4$, so all but finitely many of the integral points we find will produce solutions with c in the correct range.

Eq. (11) is easily seen to be equivalent to the norm equation

$$N(x + y\sqrt{2}) = -1$$

where N is the norm from $\mathbb{Q}(\sqrt{2})$ to \mathbb{Q} . Let $u = 1 + \sqrt{2}$ be the fundamental unit of $\mathbb{Q}(\sqrt{2})$. Note that $N(u^k) = -1$ for all odd integers k . If we let $u^k = a_k + b_k\sqrt{2}$ for integers a_k, b_k , then the point (a_k, b_k) is an integral point on Eq. (11). So we have an infinite family of such points, and thus infinitely many solutions to Eq. (10). □

3.6 The case $(q, n, \ell) = (3, 3, 1)$

Theorem 18. *There are infinitely many solutions to the case $(q, n, \ell) = (3, 3, 1)$.*

Proof. We show that the equation

$$343y^2 = x^2 + x + 1 \quad (12)$$

has infinitely many solutions in integers with $x > y > 0$. It follows that the equation

$$(7y)^3 = c(b^2 + b + 1) \quad (13)$$

has infinitely many solutions with $c = y$, $b = x$, and $1 \leq c < b$.

Completing the square on the RHS of Eq. (12), multiplying both sides by 4, and rearranging, we obtain the equivalent equation

$$((2x+1)^2 - (14y)^2(7)) = -3.$$

We write this as $N((2x+1) + 14y\sqrt{7}) = -3$, where N is the norm from $\mathbb{Q}(\sqrt{7})$ to \mathbb{Q} . Let $\alpha = a + b\sqrt{7}$ with positive integers a, b . If $N(\alpha) = -3$, a is odd, and b is divisible by 14, then $x = (a-1)/2$ and $y = b/14$ yield a solution to Eq. (12).

As in the previous theorems, we start with a single element with the desired properties, in this case $\alpha = 37 + 98\sqrt{7}$, and use the unit group to produce infinitely many. The fundamental unit of $\mathbb{Q}(\sqrt{7})$ is $u = 8 - 3\sqrt{7}$, which satisfies $u^{14} \equiv 1 \pmod{14\mathbb{Z}[\sqrt{7}]}$. Thus, any of the elements αu^{14k} will be of the desired form, and there are infinitely many solutions to Eq. (13) as well. \square

3.7 The case $(q, n, \ell) = (3, 2, 3)$

The solutions we found to the $(3, 3, 1)$ case also produce solutions to the $(3, 2, 3)$ case by a straightforward algebraic manipulation. Recall that for the $(3, 2, 3)$ case, the equation to solve is

$$y^3 = c(b^3 + 1) \tag{14}$$

with $b^2 \leq c < b^3$.

Theorem 19. *There are infinitely many solutions to the case $(q, n, \ell) = (3, 2, 3)$.*

Proof. Let (y, b, c) be a solution to the case $(q, n, \ell) = (3, 3, 1)$. Then $y^3 = c(b^2 + b + 1)$ for some integer c satisfying $1 \leq c < b$. Set $b' = b + 1$, $y' = y(b + 2)$, and $c' = c(b + 2)^2$. We claim that (y', b', c') is a solution to Eq. (14). We compute

$$(y')^3 = y^3(b + 2)^3 = c(b^2 + b + 1)(b + 2)^3 = c(b + 2)^2((b + 1)^3 + 1) = c'((b')^3 + 1),$$

as claimed. The only thing left to check is that c' is in the correct range $(b')^2 \leq c' < (b')^3$.

As $1 \leq c \leq b - 1$ by assumption, we have

$$(b + 2)^2 \leq c(b + 2)^2 \leq (b - 1)(b + 2)^2.$$

So $c' \geq (b + 2)^2 \geq (b + 1)^2 = (b')^2$, and the lower bound on c' is satisfied. For the upper bound, we directly compute

$$(b')^3 - (b - 1)(b + 2)^2 = (b + 1)^3 - (b - 1)(b + 2)^2 = 3b + 5,$$

and $3b + 5 > 0$ for all b under consideration. So $(b - 1)(b + 2)^2 < (b')^3$; thus $c' < (b')^3$, and c' is in the correct range.

We have shown there are infinitely many solutions to the $(3, 3, 1)$ case, so it follows that there are infinitely many solutions to the $(3, 2, 3)$ case. Note that $w = (c, 2c, c, 2c, c, 2c)$ in base $b' = b + 1$. \square

3.8 The case $(q, n, \ell) = (2, 4, 1)$

Here we will show that the equation

$$(y^2)_b = w \uparrow 4$$

has solutions for infinitely many bases b .

Theorem 20. *There are infinitely many solutions to the case $(q, n, \ell) = (2, 4, 1)$.*

Proof. The $(2, 4, 1)$ case requires solving the equation

$$y^2 = c(b^3 + b^2 + b + 1) \quad (15)$$

for $1 \leq c < b$. As in the $(3, 2, 2)$ case, we use the infinitely many integral points on the curve

$$2y^2 = x^2 + 1. \quad (16)$$

Notice that any such x is odd, and that we may assume that $x, y > 0$ without loss of generality.

Setting $b = x$ and $c = \frac{1}{2}(x + 1)$, and multiplying both sides of Eq. (16) by $\frac{1}{2}(x + 1)^2$, we obtain

$$(y(x + 1))^2 = \frac{1}{2}(x + 1)(x + 1)(x^2 + 1) = c(b^3 + b^2 + b + 1),$$

which gives a solution to Eq. (15). If $x > 1$, we have $1 \leq c < x$, as required. \square

3.9 The case $(q, n, \ell) = (4, 2, 2)$

Theorem 21. *There are infinitely many solutions to the $(q, n, \ell) = (4, 2, 2)$ case.*

Proof. The key equation to solve for the $(4, 2, 2)$ case is

$$y^4 = c(b^2 + 1) \quad (17)$$

for $b \leq c < b^2$. We begin as in the $(3, 2, 2)$ case by finding infinitely many integral points on the curve

$$2y^2 = x^2 + 1, \quad (18)$$

but now also insisting that y be divisible by 13. Assuming this is possible for the moment, set $b = x$ and $c = 2^3 \cdot 3^4 \cdot 13^{-4} \cdot y^2 = 2^3 \cdot 3^4 \cdot 13^{-4} \cdot (x^2 + 1)$, and note that c is an integer. Clearly $c \leq x^2 - 1$; on the other hand, $c \geq x$ holds for all $x \geq 89$ and thus for all but finitely many of the integral solutions to Eq. (18). Multiplying through by $(6y/13)^2$, we have

$$2 \left(\frac{6y}{13} \right)^4 = \frac{36}{169} y^2 (x^2 + 1) = 2c(x^2 + 1),$$

so there are infinitely many solutions to Eq. (17).

It remains to demonstrate the existence of an infinite family of integral points (x, y) on the curve Eq. (18) such that $13 \mid y$. As in the $(3, 2, 2)$ case, the equation defining the curve can be rewritten $N(x + y\sqrt{2}) = -1$ where N is the norm from $\mathbb{Q}(\sqrt{2})$ to \mathbb{Q} . Let $u = 1 + \sqrt{2}$ be the fundamental unit in $\mathbb{Q}(\sqrt{2})$. We compute $u^7 = 239 + 169\sqrt{2}$. Writing $u^{7k} = a_k + b_k\sqrt{2}$ for integers a_k, b_k , it is easy to see that $13 \mid b_k$ for all $k \geq 1$, and $N(u^{7k}) = -1$ for all odd k . So the family u^{14k+7} gives an infinite supply of points of the desired form. \square

We have now considered all admissible triples, and this completes the proof of Theorem 2. \square

4 Solutions for inadmissible triples

As we have seen above, the abc conjecture implies that there are only a finite number of solutions, in toto, corresponding to all inadmissible triples, to the equation $(y^q)_b = w \uparrow n$ with $|w| = \ell$. So far we have found 8 such solutions, and they are given below in Table 4.

| q | n | ℓ | b | y | w |
|-----|-----|--------|-------|-------------|--------------------------|
| 2 | 5 | 1 | 3 | 11 | (1) |
| 4 | 3 | 1 | 18 | 7 | (7) |
| 4 | 2 | 3 | 19 | 70 | (9, 13, 4) |
| 4 | 2 | 3 | 23 | 78 | (5, 17, 6) |
| 5 | 2 | 2 | 239 | 52 | (27, 203) |
| 5 | 2 | 2 | 239 | 78 | (211, 115) |
| 6 | 2 | 2 | 239 | 26 | (22, 150) |
| 3 | 2 | 4 | 12400 | 57459558593 | (4208, 7128, 8441, 5457) |

We searched various ranges for other solutions and our search results are summarized below.

| q | n | ℓ | b |
|-----|-----|--------|---------------------|
| 2 | 3 | 3 | none ≤ 4315000 |
| 2 | 3 | 4 | none ≤ 544200 |
| 2 | 3 | 5 | none ≤ 544200 |
| 2 | 4 | 2 | none ≤ 544200 |
| 2 | 4 | 3 | none ≤ 544200 |
| 2 | 4 | 4 | none ≤ 544200 |
| 2 | 4 | 5 | none ≤ 544200 |
| 2 | 5 | 1 | one $\leq 10^7$ |
| 2 | 5 | 2 | none ≤ 544200 |
| 2 | 5 | 3 | none ≤ 544200 |
| 2 | 6 | 1 | none ≤ 544200 |
| 2 | 6 | 2 | none ≤ 544200 |
| 2 | 6 | 3 | none ≤ 544200 |
| 3 | 2 | 4 | one $\leq 10^7$ |
| 3 | 2 | 5 | none ≤ 544200 |
| 3 | 2 | 6 | none ≤ 544200 |
| 3 | 3 | 2 | none ≤ 544200 |
| 3 | 3 | 3 | none ≤ 4315000 |
| 3 | 3 | 4 | none ≤ 544200 |
| 3 | 3 | 5 | none ≤ 544200 |
| 3 | 4 | 1 | none ≤ 544200 |
| 3 | 4 | 2 | none ≤ 544200 |
| 3 | 4 | 3 | none ≤ 544200 |
| 3 | 4 | 4 | none ≤ 544200 |
| 3 | 4 | 5 | none ≤ 544200 |
| 3 | 5 | 1 | none ≤ 544200 |
| 3 | 5 | 2 | none ≤ 544200 |
| 3 | 5 | 3 | none ≤ 544200 |
| 4 | 2 | 3 | two $\leq 10^7$ |
| 4 | 2 | 4 | none ≤ 544200 |
| 4 | 2 | 5 | none ≤ 544200 |
| 4 | 2 | 6 | none ≤ 544200 |
| 4 | 3 | 1 | one $\leq 10^7$ |
| 4 | 3 | 2 | none ≤ 544200 |
| 4 | 3 | 3 | none ≤ 4315000 |
| 4 | 3 | 4 | none ≤ 544200 |
| 4 | 3 | 5 | none ≤ 544200 |
| 4 | 4 | 1 | none ≤ 544200 |
| 4 | 4 | 2 | none ≤ 544200 |
| 4 | 4 | 3 | none ≤ 544200 |
| 4 | 4 | 4 | none ≤ 544200 |
| 4 | 4 | 5 | none ≤ 544200 |
| 4 | 5 | 1 | none ≤ 544200 |
| 4 | 5 | 2 | none ≤ 544200 |
| 4 | 5 | 3 | none ≤ 544200 |
| 5 | 2 | 2 | one $\leq 10^7$ |
| 5 | 2 | 3 | none ≤ 544200 |
| 5 | 3 | 1 | none ≤ 544200 |
| 5 | 3 | 2 | none ≤ 544200 |
| 5 | 3 | 3 | none ≤ 4315000 |
| 5 | 4 | 1 | none ≤ 544200 |
| 5 | 4 | 2 | none ≤ 544200 |
| 5 | 4 | 3 | none ≤ 544200 |
| 6 | 2 | 2 | one $\leq 10^7$ |
| 6 | 2 | 3 | none ≤ 544200 |
| 6 | 3 | 1 | none ≤ 544200 |
| 6 | 3 | 2 | none ≤ 544200 |
| 6 | 3 | 3 | none ≤ 4315000 |
| 6 | 4 | 1 | none ≤ 544200 |

4.1 Our search procedure

Consider Eq. (3): $y^q = c \frac{b^{n\ell} - 1}{b^\ell - 1}$. We describe a search procedure to find solutions (b, y) to this equation, which produced the results above. It has been implemented in three different languages: APL, Maple, and python. Code is available from the authors.

Given (q, n, ℓ) and b , we start by factoring $r := (b^{n\ell} - 1)/(b^\ell - 1)$. This prime factorization can be speeded up using the algebraic factorization of the polynomial $X^{(n-1)\ell} + \dots + X^\ell + 1$ over $\mathbb{Q}[X]$. For example, if $n = 3$ and $\ell = 2$, the polynomial $X^4 + X^2 + 1$ has the factorization $f(X) \cdot g(X)$ where $f(X) = X^2 + X + 1$ and $g(X) = X^2 - X + 1$. We therefore can compute $f(b)$ and $g(b)$ and factor each piece independently and combine the results.

Now we have the prime factorization of r , say $r = p_1^{e_1} \cdots p_t^{e_t}$. If cr is to be a q th power, then we must have that $p_i^{q[e_i/q]}$ divides cr for $1 \leq i \leq t$. So c must be a multiple of

$$d := \prod_{1 \leq i \leq t} p_i^{q[e_i/q] - e_i}.$$

and c must further satisfy the inequality $b^{\ell-1} \leq c < b^\ell$. Writing $c = k^q d$ for some integer k ,

we have $(b^\ell - 1)/d \leq k^q < b^\ell/d$ and so $((b^\ell - 1)/d)^{1/q} \leq k < (b^\ell/d)^{1/q}$. A solution then exists for each integer k in this interval, which can be easily checked.

The most time-consuming part of this calculation is the integer factorization. Typically we were searching some subrange of the interval $[2, 10^7]$, with $n \leq 6$ and $\ell \leq 5$. Thus we could be factoring numbers of size as large as 10^{175} . If we want to perform this computation for many different triples (q, n, ℓ) at once, it makes sense to first precompute the algebraic factorizations described above, next compute the factorizations of individual pieces, and finally assemble the needed factorizations from these pieces.

5 Beyond canonical base- b representation

One can consider the equation $(y^q) = w \uparrow n$ for a wide variety of other types of representations. In this section we consider two such other types of representations.

5.1 Bijective base- b representation

First, we consider the so-called “bijective base- b representation”; see, for example, [19]; [46, §9, pp. 34–36]; [25, Solution to Exercise 4.1-24, p. 495]; [43, Note 9.1, pp. 90–91]; [17, pp. 70–76]; [18]; [4].

This representation is like ordinary base- b representation, except that instead of using the digits $0, 1, \dots, b - 1$, we use the digits $1, 2, \dots, b$ instead. We use the notation $\langle x \rangle_b$ to denote this representation.

Theorem 22. *For all $b, \ell \geq 2$ there exists a word w of length ℓ and an integer y such that $\langle y^2 \rangle_b = w \uparrow 2$.*

Proof. Consider $y = b^\ell + 1$ for $\ell \geq 2$. Then $y^2 = b^{2\ell} + 2b^\ell + 1$, which has bijective base- b representation $w \uparrow 2$ for $w = ((b - 1) \uparrow (\ell - 2), b, 1)$. \square

For some specific bases b there are other infinite families of solutions. For example, the table below summarizes some of these families, for $n \geq 0$. They are easy to prove by direct calculation.

| b | $\langle y \rangle_b$ | $\langle y^2 \rangle_b$ |
|-----|----------------------------|---|
| 2 | $((12) \uparrow 3n+3)212$ | $((((221112) \uparrow n)221121112) \uparrow 2$ |
| 3 | $((1331) \uparrow 5n+2)22$ | $((12132111223231233322) \uparrow n)1213211131) \uparrow 2$ |
| 4 | $((21) \uparrow 5n+2)3$ | $((((1123421433) \uparrow n)11241) \uparrow 2$ |
| 4 | $((24) \uparrow 5n+2)4$ | $((((2143311234) \uparrow n)21434) \uparrow 2$ |
| 5 | $((31) \uparrow 3n+1)4$ | $((((155234) \uparrow n)211) \uparrow 2$ |
| 6 | $((41) \uparrow 7n+3)5$ | $((((26211162534435) \uparrow n)2621121) \uparrow 2$ |
| 6 | $((46) \uparrow 7n+3)6$ | $((((42236551331456) \uparrow n)4223656) \uparrow 2$ |
| 7 | $(3 \uparrow 2n+2)4$ | $((((15) \uparrow n+1)2) \uparrow 2$ |
| 8 | $((52) \uparrow n+1)6$ | $((((34) \uparrow n+1)4) \uparrow 2$ |
| 9 | $((35) \uparrow 10n+2)4$ | $((((1385674932) \uparrow 2n)13857) \uparrow 2$ |
| 9 | $((53) \uparrow 10n+2)6$ | $((((3213856749) \uparrow 2n)32139) \uparrow 2$ |
| 9 | $((71) \uparrow 10n+2)8$ | $((((5674932138) \uparrow 2n)56751) \uparrow 2$ |

5.2 Fibonacci representation

Yet another representation for integers involves the Fibonacci numbers. The so-called Fibonacci or Zeckendorf representation of an integer $n \geq 0$ consists of writing n as the sum of non-adjacent Fibonacci numbers:

$$n = \sum_{2 \leq i \leq t} e_i F_i$$

where $e_i \in \{0, 1\}$ and $e_i e_{i+1} \neq 1$ for $i \geq 2$; see [29, 51]. In this case we write the representation of n , starting with the most significant digit, as the binary word $(n)_F = e_t e_{t-1} \cdots e_2$.

Theorem 23. *There are infinitely many solutions to the equation $(y^2)_F = w \uparrow 2$, for integers y and words w .*

Proof. The proof depends on the following identity:

$$\begin{aligned} (F_{4n+3} + F_{4n+6} + F_{8n+8} + F_{8n+11})^2 = \\ F_{4n+2} + F_{4n+5} + F_{4n+8} + F_{4n+10} + \left(\sum_{1 \leq i < n} F_{4n+4i+10} \right) + F_{8n+11} + \\ F_{12n+12} + F_{12n+15} + F_{12n+18} + F_{12n+20} + \left(\sum_{1 \leq i < n} F_{12n+4i+20} \right) + F_{16n+21}, \end{aligned}$$

which can be proved with a computer algebra system, such as Maple.

This identity shows that the Fibonacci representation of $(F_{4n+3} + F_{4n+6} + F_{8n+8} + F_{8n+11})^2$ has the form w^2 with

$$w = (1, 0, 0, 0, 0, ((1, 0, 0, 0) \uparrow (n-1)), 1, 0, 1, 0, 0, 1, 0, 0, 1, (0 \uparrow (4n))).$$

□

Remark 24. There are other infinite families of solutions. Here is one: let $n \geq 1$ and suppose $(y)_F = ((100) \uparrow (4n + 2), 1, 0, 1, 0, 0, 0)$. Then $(y^2)_F = ww$ with $w = (((1, 0, 0, 1, (0 \uparrow 8)) \uparrow n), 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0)$.

A list of all the solutions to $(y^2)_F = w \uparrow 2$ with $y < 34000000$ is given below.

| <i>y</i> | <i>w</i> |
|----------|---------------------------------------|
| 4 | 100 |
| 49 | 10100100 |
| 306 | 10010000010 |
| 728 | 10000000101000 |
| 2021 | 1000100000101010 |
| 3556 | 10010101001000100 |
| 3740 | 10100101001000010 |
| 5236 | 100001010010010000 |
| 21360 | 100000010100101010010 |
| 35244 | 1000010000001010000000 |
| 98210 | 100100000000100100000010 |
| 243252 | 10000100010100100100000000 |
| 1096099 | 10010000010100100100010101010 |
| 1625040 | 100000010101001010000100001000 |
| 1662860 | 100001000100000010100000000000 |
| 4976785 | 1010001000010000000000000010100100 |
| 5080514 | 101001001010010000000001000010100 |
| 11408968 | 10000100010001010010010000000000000 |
| 31622994 | 1001000000001001000000000100100000010 |
| 31831002 | 100100000101010000010000010101000010 |
| 33587514 | 101000000100100010001000100101000000 |
| 33599070 | 101000000100101001000010000101000000 |

Remark 25. The only solutions to $(y^q)_F = w \uparrow n$ other than $(q, n) = (2, 2)$ that we found are $(2^4)_F = (100) \uparrow 2$ and $(7^4)_F = (10100100) \uparrow 2$.

References

- [1] M. Bennett. Rational approximation to algebraic numbers of small height: the Diophantine equation $|ax^n - by^n| = 1$. *J. Reine Angew. Math.* **535** (2001), 1–49.
- [2] M. Bennett and A. Levin. The Nagell-Ljunggren equation via Runge’s method. *Monatsh. Math.* **177** (2015), 15–31.

- [3] J. Berstel. *Axel Thue's Papers on Repetitions in Words: a Translation*. Number 20 in Publications du Laboratoire de Combinatoire et d'Informatique Mathématique. Université du Québec à Montréal, February 1995.
- [4] R. T. Boute. Zeroless positional number representation and string ordering. *Amer. Math. Monthly* **107** (2000), 437–444.
- [5] K. A. Broughan. An explicit bound for aliquot cycles of repdigits. *INTEGERS* **12** (2012), #A15 (electronic).
- [6] J. Browkin. The *abc*-conjecture. In *Number Theory*, Trends Math., pp. 75–105. Birkhäuser, 2000.
- [7] J. Browkin. A weak effective *abc*-conjecture. *Funct. Approx. Comment. Math.* **39** (2008), 103–111.
- [8] Y. Bugeaud. Linear forms in two m -adic logarithms and applications to diophantine problems. *Compositio Math.* **132** (2002), 137–158.
- [9] Y. Bugeaud, G. Hanrot, and M. Mignotte. Sur l'équation diophantienne $(x^n - 1)/(x - 1) = y^q$. III. *Proc. Lond. Math. Soc.* **84** (2002), 59–78.
- [10] Y. Bugeaud and M. Mignotte. On integers with identical digits. *Mathematika* **46** (1999), 411–417.
- [11] Y. Bugeaud and M. Mignotte. Sur l'équation diophantienne $(x^n - 1)/(x - 1) = y^q$. II. *C. R. Acad. Sci. Paris* **328** (1999), 741–744.
- [12] Y. Bugeaud and M. Mignotte. L'équation de Nagell-Ljunggren $\frac{x^n - 1}{x - 1} = y^q$. *Enseign. Math.* **48** (2002), 147–168.
- [13] Y. Bugeaud, M. Mignotte, and Y. Roy. On the diophantine equation $(x^n - 1)/(x - 1) = y^q$. *Pacific J. Math.* **193** (2000), 257–268.
- [14] Y. Bugeaud, M. Mignotte, Y. Roy, and T. N. Shorey. The equation $(x^n - 1)/(x - 1) = y^q$ has no solution with x square. *Math. Proc. Cambridge Phil. Soc.* **127** (1999), 353–372.
- [15] Y. Bugeaud and P. Mihăilescu. On the Nagell-Ljunggren equation $\frac{x^n - 1}{x - 1} = y^q$. *Math. Scand.* **101** (2007), 177–183.
- [16] J. Cilleruelo, F. Luca, and I. E. Shparlinski. Power values of palindromes. *J. Combin. Number Theory* **1** (2009), 101–107.
- [17] M. D. Davis and E. J. Weyuker. *Computability, Complexity, and Languages: Fundamentals of Theoretical Computer Science*. Academic Press, 1983.

- [18] R. R. Forsslund. A logical alternative to the existing positional number system. *Southwest J. Pure Appl. Math.* **1** (1995), 27–29.
- [19] J. E. Foster. A number system without a zero symbol. *Math. Mag.* **21**(1) (1947), 39–41.
- [20] A. Granville. Rational and integral points on quadratic twists of a given hyperelliptic curve. *Int. Math. Res. Not. IMRN*, No. 8, (2007), Art. ID 027.
- [21] A. Granville and T. J. Tucker. It's as easy as abc . *Notices Amer. Math. Soc.* **49** (2002), 1224–1231.
- [22] S. Hernández and F. Luca. Palindromic powers. *Rev. Colombiana Mat.* **40** (2006), 81–86.
- [23] N. Hirata-Kohno and T. N. Shorey. On the equation $(x^m - 1)/(x - 1) = y^q$ with x power. In Y. Motohashi, editor, *Analytic number theory*, Vol. 247 of *London Math. Soc. Lecture Note Series*, pp. 343–351. Cambridge University Press, 1997.
- [24] O. Kihel. A note on the Nagell-Ljunggren Diophantine equation $\frac{x^n - 1}{x - 1} = y^q$. *JP J. Algebra Number Theory Appl.* **13** (2009), 131–135.
- [25] D. E. Knuth. *The Art of Computer Programming. Volume 2: Seminumerical Algorithms*. Addison-Wesley, 1969. 1st edition.
- [26] I. Korec. Palindromic squares for various number system bases. *Math. Slovaca* **41** (1991), 261–276.
- [27] S. Laishram and T. N. Shorey. Baker's explicit abc -conjecture and applications. *Acta Arith.* **155** (2012), 419–429.
- [28] M. H. Le. A note on the diophantine equation $\frac{x^m - 1}{x - 1} = y^n$. *Math. Proc. Cambridge Phil. Soc.* **116** (1994), 385–389.
- [29] C. G. Lekkerkerker. Voorstelling van natuurlijke getallen door een som van getallen van Fibonacci. *Simon Stevin* **29** (1952), 190–195.
- [30] J. Li and X. Li. On the Nagell-Ljunggren equation and Edgar's conjecture. *Int. J. Appl. Math. Stat.* **52** (2014), 80–83.
- [31] W. Ljunggren. Einige Bemerkungen über die Darstellung ganzer Zahlen durch binäre kubische Formen mit positiver Diskriminante. *Acta Math.* **75** (1943), 1–21.
- [32] W. Ljunggren. Noen setninger om ubestemte likninger av formen $(x^n - 1)/(x - 1) = y^q$. *Norsk. Mat. Tidsskr.* **25** (1943), 17–20.

- [33] D. W. Masser. Problem. In *Symposium on Analytic Number Theory in Honour of K. F. Roth*, p. 25. Department of Mathematics, Imperial College, London, 1985. Book of abstracts of conference.
- [34] P. Mihăilescu. New bounds and conditions for the equation of Nagell-Ljunggren. *J. Number Theory* **124** (2007), 380–395.
- [35] P. Mihăilescu. Class number conditions for the diagonal case of the equation of Nagell and Ljunggren. In *Diophantine approximation*, Vol. 16 of *Dev. Math.*, pp. 245–273. Springer-Verlag, 2008.
- [36] T. Nagell. Note sur l'équation indéterminée $(x^n - 1)/(x - 1) = y^q$. *Norsk. Mat. Tidsskr.* **2** (1920), 75–78.
- [37] T. Nagell. Des équations indéterminées $x^2 + x + 1 = y^n$ et $x^2 + x + 1 = 3y^n$. *Norsk Mat. Forenings Skrifter*, 1921.
- [38] A. Nitaj. La conjecture abc . *Enseign. Math.* **42** (1996), 3–24.
- [39] R. Obláth. Une propriété des puissances parfaites. *Mathesis* **65** (1956), 356–364.
- [40] N. J. A. Sloane et al. The on-line encyclopedia of integer sequences. Available at <https://oeis.org>.
- [41] Joseph Oesterlé. Nouvelles approches du “théorème” de Fermat. *Astérisque* **161-162** (1988), 165–186. Séminaire Bourbaki, exp. 694.
- [42] O. Robert, C. Stewart, and G. Tenenbaum. A refinement of the abc conjecture. *Bull. Lond. Math. Soc.* **46** (2014), 1156–1166.
- [43] A. Salomaa. *Formal Languages*. Academic Press, 1973.
- [44] T. N. Shorey. On the equation $z^q = (x^n - 1)/(x - 1)$. *Indag. Math.* **48** (1986), 345–351.
- [45] T. N. Shorey. Some conjectures in the theory of exponential diophantine equations. *Publ. Math. (Debrecen)* **56** (2000), 631–641.
- [46] R. M. Smullyan. *Theory of Formal Systems*, Vol. 47 of *Annals of Mathematical Studies*. Princeton University Press, 1961.
- [47] C. L. Stewart and R. Tijdeman. On the Oesterlé-Masser conjecture. *Monatsh. Math.* **102** (1986), 251–257.
- [48] A. Thue. Über unendliche Zeichenreihen. *Norske vid. Selsk. Skr. Mat. Nat. Kl.* **7** (1906), 1–22. Reprinted in *Selected Mathematical Papers of Axel Thue*, T. Nagell, editor, Universitetsforlaget, Oslo, 1977, pp. 139–158.

[49] A. Thue. Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen. *Norske vid. Selsk. Skr. Mat. Nat. Kl.* **1** (1912), 1–67. Reprinted in *Selected Mathematical Papers of Axel Thue*, T. Nagell, editor, Universitetsforlaget, Oslo, 1977, pp. 413–478.

[50] S. Yates. The mystique of repunits. *Math. Mag.* **51** (1978), 22–28.

[51] E. Zeckendorf. Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres Lucas. *Bull. Soc. Roy. Liége* **41** (1972), 179–182.

A Tables of solutions for admissible triples

Here we list some of the smaller solutions to Eq. (2) corresponding to admissible triples.

| b | y | w |
|-----|------|-------|
| 18 | 49 | (7) |
| 22 | 39 | (3) |
| 22 | 78 | (12) |
| 30 | 133 | (19) |
| 68 | 247 | (13) |
| 68 | 494 | (52) |
| 146 | 1651 | (127) |
| 292 | 4503 | (237) |
| 313 | 543 | (3) |
| 313 | 1086 | (12) |
| 313 | 1629 | (27) |
| 313 | 2172 | (48) |
| 313 | 2715 | (75) |
| 313 | 3258 | (108) |
| 313 | 3801 | (147) |
| 313 | 4344 | (192) |
| 313 | 4887 | (243) |
| 313 | 5430 | (300) |
| 423 | 4171 | (97) |
| 423 | 8342 | (388) |
| 439 | 6231 | (201) |
| 499 | 2289 | (21) |
| 499 | 4578 | (84) |
| 499 | 6867 | (189) |
| 499 | 9156 | (336) |

Table 1: Solutions to $(q, n, \ell) = (2, 3, 1)$ with $b \leq 500$

| b | y | w |
|---------|----------------------|--------------------|
| 68 | 160797 | (17, 53) |
| 313 | 7575393 | (19, 32) |
| 313 | 15150786 | (76, 128) |
| 313 | 22726179 | (171, 288) |
| 313 | 30301572 | (305, 199) |
| 699 | 274088893 | (450, 133) |
| 4366 | 4649437443 | (13, 2735) |
| 4366 | 9298874886 | (54, 2208) |
| 4366 | 13948312329 | (122, 2785) |
| 4366 | 18597749772 | (218, 100) |
| 4366 | 23247187215 | (340, 2885) |
| 4366 | 27896624658 | (490, 2408) |
| 4366 | 32546062101 | (667, 3035) |
| 4366 | 37195499544 | (872, 400) |
| 4366 | 41844936987 | (1103, 3235) |
| 4366 | 46494374430 | (1362, 2808) |
| 4366 | 51143811873 | (1648, 3485) |
| 4366 | 55793249316 | (1962, 900) |
| 4366 | 60442686759 | (2302, 3785) |
| 4366 | 65092124202 | (2670, 3408) |
| 4366 | 69741561645 | (3065, 4135) |
| 4366 | 74390999088 | (3488, 1600) |
| 4366 | 79040436531 | (3938, 169) |
| 51567 | 134669813878873 | (49737, 9460) |
| 234924 | 3266513519259697 | (14911, 203389) |
| 234924 | 6533027038519394 | (59647, 108784) |
| 234924 | 9799540557779091 | (134206, 186033) |
| 686287 | 187720229347705587 | (231469, 166496) |
| 3526450 | 27308133273274738941 | (1367399, 1295431) |
| 3652434 | 40207131048785611981 | (2487105, 3465907) |

Table 2: Solutions to $(q, n, \ell) = (2, 3, 2)$ with $b \leq 5 \cdot 10^6$

| b | y | w |
|-----|------|------------|
| 7 | 10 | (2, 6) |
| 38 | 85 | (11, 7) |
| 41 | 58 | (2, 34) |
| 41 | 116 | (22, 26) |
| 57 | 130 | (11, 49) |
| 68 | 185 | (20, 9) |
| 117 | 370 | (31, 73) |
| 239 | 338 | (2, 198) |
| 239 | 676 | (22, 150) |
| 239 | 1014 | (76, 88) |
| 239 | 1352 | (181, 5) |
| 268 | 1105 | (70, 25) |
| 515 | 2626 | (132, 296) |
| 682 | 915 | (2, 283) |
| 682 | 1220 | (5, 494) |
| 682 | 1525 | (11, 123) |
| 682 | 1830 | (19, 218) |
| 682 | 2135 | (30, 463) |
| 682 | 2440 | (45, 542) |
| 682 | 2745 | (65, 139) |
| 682 | 3050 | (89, 302) |
| 682 | 3355 | (119, 33) |
| 682 | 3660 | (154, 380) |
| 682 | 3965 | (196, 345) |
| 682 | 4270 | (245, 294) |
| 682 | 4575 | (301, 593) |
| 682 | 4880 | (366, 244) |
| 682 | 5185 | (439, 295) |
| 682 | 5490 | (521, 430) |
| 682 | 5795 | (613, 333) |
| 882 | 5365 | (225, 55) |

Table 3: Solutions to $(q, n, \ell) = (3, 2, 2)$ with $b \leq 1000$

| b | y | w |
|---------|---------|-----------|
| 18 | 7 | (1) |
| 18 | 14 | (8) |
| 88916 | 24661 | (1897) |
| 88916 | 49322 | (15176) |
| 88916 | 73983 | (51219) |
| 1147805 | 631111 | (190801) |
| 6042955 | 3956043 | (1695447) |

Table 4: Solutions to $(q, n, \ell) = (3, 3, 1)$ with $b \leq 10^7$

| b | y | w |
|-----|--------|-----------------|
| 8 | 57 | (5, 5, 1) |
| 19 | 140 | (1, 2, 1) |
| 19 | 210 | (3, 14, 1) |
| 19 | 280 | (8, 16, 8) |
| 19 | 350 | (17, 5, 18) |
| 23 | 234 | (1, 22, 18) |
| 23 | 312 | (4, 16, 12) |
| 23 | 390 | (9, 4, 22) |
| 23 | 468 | (15, 21, 6) |
| 31 | 532 | (5, 8, 1) |
| 80 | 2709 | (6, 5, 29) |
| 80 | 5418 | (48, 42, 72) |
| 215 | 39438 | (133, 112, 42) |
| 293 | 63042 | (116, 7, 101) |
| 314 | 19005 | (2, 78, 41) |
| 314 | 38010 | (17, 311, 14) |
| 314 | 57015 | (60, 225, 165) |
| 314 | 76020 | (143, 290, 112) |
| 314 | 95025 | (281, 32, 101) |
| 362 | 29337 | (4, 22, 117) |
| 362 | 58674 | (32, 178, 212) |
| 362 | 88011 | (109, 240, 263) |
| 362 | 117348 | (259, 342, 248) |
| 374 | 99645 | (135, 78, 189) |
| 440 | 43617 | (5, 13, 393) |
| 440 | 87234 | (40, 111, 64) |
| 440 | 130851 | (135, 375, 51) |
| 440 | 174468 | (322, 9, 72) |
| 485 | 108342 | (47, 188, 433) |
| 485 | 216684 | (379, 56, 69) |

Table 5: Solutions to $(q, n, \ell) = (3, 2, 3)$ with $b \leq 1000$

| b | y | w |
|--------|-------------|---------|
| 7 | 20 | (1) |
| 7 | 40 | (4) |
| 41 | 1218 | (21) |
| 99 | 7540 | (58) |
| 239 | 20280 | (30) |
| 239 | 40560 | (120) |
| 1393 | 1373090 | (697) |
| 2943 | 4903600 | (943) |
| 8119 | 23308460 | (1015) |
| 8119 | 46616920 | (4060) |
| 45368 | 316540365 | (1073) |
| 45368 | 633080730 | (4292) |
| 45368 | 949621095 | (9657) |
| 45368 | 1266161460 | (17168) |
| 45368 | 1582701825 | (26825) |
| 45368 | 1899242190 | (38628) |
| 47321 | 527813814 | (2629) |
| 47321 | 1055627628 | (10516) |
| 47321 | 1583441442 | (23661) |
| 47321 | 2111255256 | (42064) |
| 82417 | 4091661910 | (29905) |
| 144721 | 16718044954 | (92209) |

Table 6: Solutions to $(q, n, \ell) = (2, 4, 1)$ with $b \leq 2.5 \cdot 10^5$

| b | y | w |
|----------|----------|----------------------|
| 239 | 78 | (2, 170) |
| 239 | 104 | (8, 136) |
| 239 | 130 | (20, 220) |
| 239 | 156 | (43, 91) |
| 239 | 182 | (80, 88) |
| 239 | 208 | (137, 25) |
| 239 | 234 | (219, 147) |
| 682 | 305 | (27, 191) |
| 682 | 610 | (436, 328) |
| 4443 | 2810 | (710, 3910) |
| 12943 | 7930 | (1823, 10935) |
| 275807 | 78010 | (1765, 45453) |
| 275807 | 156020 | (28242, 175634) |
| 275807 | 234030 | (142978, 96202) |
| 6826318 | 386165 | (69, 6204023) |
| 6826318 | 772330 | (1118, 3695916) |
| 6826318 | 1158495 | (5662, 4204649) |
| 6826318 | 1544660 | (17896, 4524112) |
| 6826318 | 1930825 | (43693, 165751) |
| 6826318 | 2316990 | (90601, 5837522) |
| 6826318 | 2703155 | (167851, 833347) |
| 6826318 | 3089320 | (286346, 4122612) |
| 6826318 | 3475485 | (458671, 6086987) |
| 6826318 | 3861650 | (699088, 2652016) |
| 6826318 | 4247815 | (1023535, 2113435) |
| 6826318 | 4633980 | (1449629, 4658218) |
| 6826318 | 5020145 | (1996666, 2364577) |
| 6826318 | 5406310 | (2685617, 6507234) |
| 6826318 | 5792475 | (3539134, 6599513) |
| 6826318 | 6178640 | (4581545, 4524930) |
| 6826318 | 6564805 | (5838856, 884557) |
| 26392464 | 22459777 | (13841461, 10675249) |
| 30349818 | 21885565 | (8206590, 23891377) |
| 54608393 | 5940610 | (7647, 52780529) |
| 54608393 | 11881220 | (122367, 25362569) |
| 54608393 | 17821830 | (619485, 15768195) |
| 54608393 | 23762440 | (1957879, 23542353) |
| 54608393 | 29703050 | (4779979, 4361253) |
| 54608393 | 35643660 | (9911764, 33857548) |
| 54608393 | 41584270 | (18362767, 34578369) |
| 54608393 | 47524880 | (31326070, 49027290) |
| 54608393 | 53465490 | (50178308, 21230756) |
| 54610269 | 28740098 | (4189189, 40498327) |

Table 7: Solutions to $(q, n, \ell) = (4, 2, 2)$ with $b \leq 10^8$